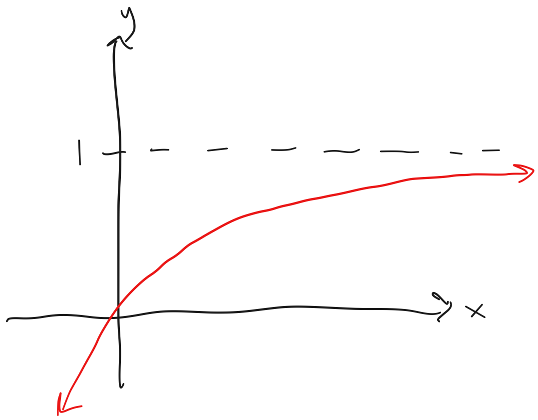
Supplementary Lecture §2.7: Limits at Infinity

+ a "bonus problem" from §2.5: Indeterminate Forms

Consider $f(x) = 1 - e^{-x}$



Notice As x gets larger, $f(x)$ gets closer to one

Write $\lim\limits_{x \to \infty} f(x) = 1$

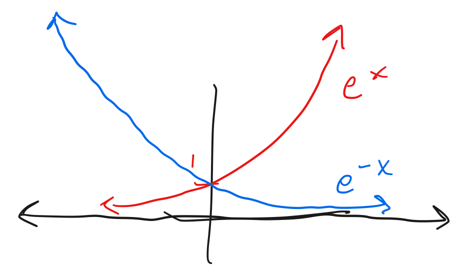
The graph $y = f(x)$ has a horizontal asymptote at $y = 1$

Note "$x \to \infty$" means "x increases w/o bound"

Examples

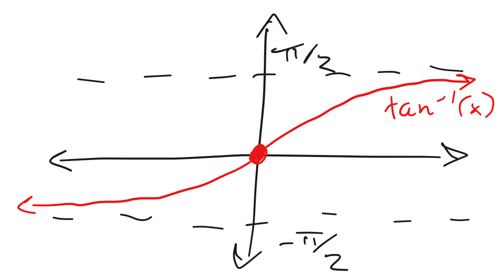
1) $\lim\limits_{x \to \infty} e^x = \infty$ "diverges to ∞"

2) $\lim\limits_{x \to \infty} e^{-x} = 0$

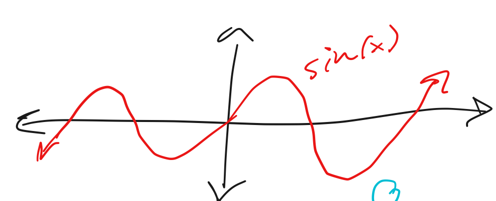


3) $\lim\limits_{x \to \infty} \underbrace{\tan^{-1}(x)}_{\arctan(x)} = \pi/2$

4) $\lim\limits_{x \to -\infty} \tan^{-1}(x) = -\pi/2$



5) $\lim\limits_{x \to \infty} \sin(x)$ DNE



Rational functions

Ex $\lim\limits_{x \to \infty} \dfrac{3x-5}{2x+3} = \lim\limits_{x \to \infty} \left[\dfrac{\left(\frac{1}{x}\right)(3x-5)}{\left(\frac{1}{x}\right)(2x-3)}\right] = \lim\limits_{x \to \infty} \left[\dfrac{3 - 5/x}{2 - 3/x}\right]$

indeteterminate form $\frac{\infty}{\infty}$

0 0

$= \boxed{3/2}$

Ex $\lim\limits_{x \to \infty} \dfrac{x^2+2}{3x^2+5x+1} = \lim\limits_{x \to \infty} \left[\dfrac{\left(\frac{1}{x^2}\right)(x^2+2)}{\left(\frac{1}{x^2}\right)(3x^2+5x+1)}\right] = \lim\limits_{x \to \infty} \left[\dfrac{1 + \cancel{2/x^2} \to 0}{3 + \cancel{5/x} + \cancel{1/x^2}}\right]$

indeterminate form $\frac{\infty}{\infty}$

$$= \boxed{1/3}$$

In general, for rational functions, look at "leading terms" for numerator and denominator. (See Thm 2 of §2.7 in book)

Ex $\displaystyle \lim_{x\to\infty} \frac{x^{3/2} - x^{1/2}}{\sqrt{4x^3+9}} \longrightarrow \frac{\text{"}\infty - \infty\text{"}}{\infty}$

To algebraically simplify $\dfrac{x^{3/2} - x^{1/2}}{\sqrt{4x^3+9}}$

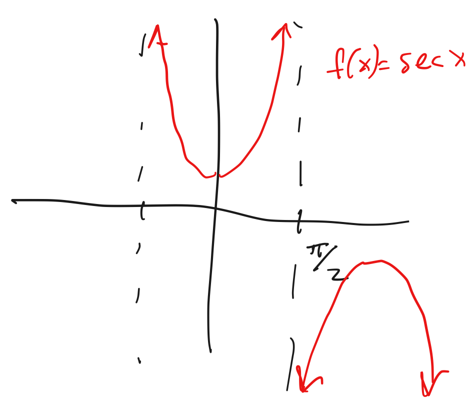
1) $x^{3/2} - x^{1/2} = x^{3/2}(1 - x^{-1}) = x^{3/2}\left(1 - \frac{1}{x}\right)$ factor out highest power of x

2) $\sqrt{4x^3+9} = \sqrt{x^3}\sqrt{4 + 9/x^3} = x^{3/2}\sqrt{4 + 9/x^2}$ "

Then, $\displaystyle \lim_{x\to\infty} \frac{x^{3/2} - x^{1/2}}{\sqrt{4x^3+9}} = \lim_{x\to\infty} \frac{\cancel{x^{3/2}}\left(1 - \cancel{\frac{1}{x}}\right)}{\cancel{x^{3/2}}\sqrt{4 + \cancel{9/x^2}}} = \frac{1}{\sqrt{4}} = \boxed{\frac{1}{2}}$

Now one extra problem from §2.5

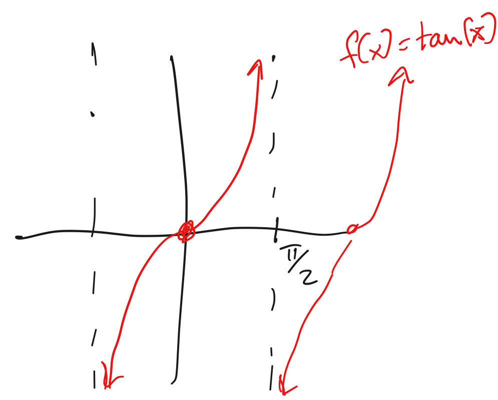
Problem $\displaystyle \lim_{\theta\to\pi/2} \left(\sec(\theta) - \tan(\theta)\right)$



$\sec(\theta) = \dfrac{1}{\cos(\theta)}$, $\displaystyle \lim_{\theta\to\pi/2^-} \sec(\theta) = +\infty$

$\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, $\displaystyle \lim_{\theta\to\pi/2^-} \tan(\theta) = +\infty$

$\displaystyle \lim_{\theta\to\pi/2} \left(\sec(\theta) - \tan(\theta)\right)$ has indeterminate form $\infty - \infty$



Manipulate algebraically:

$$\sec(\theta) - \tan(\theta) = \frac{1}{\cos(\theta)} - \frac{\sin(\theta)}{\cos(\theta)}$$

$$= \frac{1-\sin(\theta)}{\cos(\theta)} \cdot \frac{(1+\sin(\theta))}{(1+\sin(\theta))}$$

$$= \frac{1-\sin^2(\theta)}{\cos(\theta)(1+\sin(\theta))} \qquad \sin^2(\theta)+\cos^2(\theta)=1$$

$$= \frac{\cos^2(\theta)}{\cos(\theta)(1+\sin(\theta))}$$

$$= \frac{\cos(\theta)}{\cos(\theta)} \cdot \frac{\cos(\theta)}{(1+\sin(\theta))}$$

$$\lim_{\theta \to \pi/2} \left(\sec(\theta) - \tan(\theta)\right) = \lim_{\theta \to \pi/2} \left[\overset{1}{\cancel{\frac{\cos\theta}{\cos\theta}}} \cdot \frac{\cos\theta}{1+\sin\theta} \right]$$

$$= \lim_{\theta \to \pi/2} \overset{1}{\cancel{\frac{\cos\theta}{\cos\theta}}} \cdot \lim_{\theta \to \pi/2} \frac{\cos\theta}{1+\sin\theta} \qquad \text{(Product Law)}$$

$$= \frac{\cos(\pi/2)}{1+\sin(\pi/2)} = \frac{0}{1+1} = \boxed{0}$$